Deriving the Feynman Path Integral

(QFT Path Integral Formalism)

Propagator - some matrix element which evolves in time from a state $|x\rangle$ to $|x'\rangle$:

$$K(x', x, T) = \langle x' | e^{-iHT} | x\rangle$$

Probability of particle propagating from $|x\rangle \to |x'\rangle$ over some time.

$$\oint |x\rangle\langle x| = \mathbb{1} \qquad \text{(completeness)}$$

A state $\psi(x', T)$ evolves in the $|x'\rangle$ basis as

$$\psi(x', T) = \langle x' | e^{-iHT} | \psi(T=0)\rangle$$

$$= \int dx \ \langle x' | e^{-iHT} | x\rangle \langle x | \psi(0)\rangle$$

$$= \int dx \ K(x', x, T) \ \psi(x, T=0)$$

K acts as the "Green's Function" to the SE.

Insert a complete set of energy eigenfunctions:

$(\psi_n(x) = \langle n | x\rangle)$

$$K(x', x, T) = \sum_{n', n} \langle x' | n' \rangle \langle n' | e^{-iHT} | n \rangle \langle n | x \rangle$$

$$= \sum_{n', n} e^{-iE_n T} \langle x' | n' \rangle \overbrace{\cancel{\langle n' | n \rangle}}^{\delta^{nn'}} \langle n | x \rangle$$

$$= \sum_{n} e^{-iE_n T} \langle x' | n \rangle \langle n | x \rangle$$

$$= \sum_{n} e^{-iE_n T} \psi_n^{\dagger}(x') \psi_n(x)$$

Spectral Representation of Propagator

Furthermore:

$$K(x', x, T) = \langle x' | \left(e^{-iH\frac{T}{N}} \right)^N | x \rangle$$

N arbitrary, take $N \to \infty$.

$$= \int \cdots \int dx_{N-1} \cdots dx_1 \; \langle x' | e^{-iH\frac{T}{N}} | x_{N-1} \rangle$$

$$\langle x_{N-1} | e^{-iH\frac{T}{N}} | x_{N-2} \rangle \cdots \langle x_1 | e^{-iH\frac{T}{N}} | x \rangle$$

Inserting completeness $\int dx_i \, |x_i\rangle\langle x_i|$ $N-1$ times for $N-1$ orthogonal basis states.

Introduce:

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p \cdot x} \qquad T \equiv -ix \qquad \Delta\tau \equiv \frac{\tau}{N} = \frac{iT}{N} = i\Delta t$$

$(\hbar = c = 1)$

Insert completeness of momentum space:

$$K(x', x, T) = \int dx_{N-1} \dots dx_1 \int dP_{N-1} \dots dP_0 \; \langle x' | P_{N-1} \rangle$$
$$\langle P_{N-1} | e^{-iH\frac{T}{N}} | x_{N-1} \rangle \langle x_{N-1} | P_{N-2} \rangle \dots$$

Our Hamiltonian H is $H \simeq \frac{\hat{P}^2}{2m} + V(\hat{x})$.

Then $e^{A+B} = e^A e^B \implies e^{-\frac{i}{N}(A+B)} = e^{-\frac{i}{N}A} e^{-\frac{i}{N}B} \left(1 + \frac{[A,B]}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\right)$

For $\lim_{N \to \infty}$, these error correction terms vanish so we can just ignore them for now.

$$= \int dx_{N-1} \dots dx_1 \int dP_{N-1} \dots dP_0 \; \frac{1}{\sqrt{2\pi}} e^{iP_{N-1}x'} \langle P_{N-1} |$$
$$e^{-i\frac{\hat{P}^2}{2m}\frac{T}{N}} e^{-iV(\hat{x})\frac{T}{N}} | x_{N-1} \rangle \langle x_{N-1} | P_{N-2} \rangle \dots$$

Recall $e^{\hat{A}} |\alpha\rangle = \left(1 + \hat{A} + \frac{1}{2}\hat{A}^2 + \dots\right) |\alpha\rangle = e^{\alpha} |\alpha\rangle$ $\quad \left(\hat{A}|\alpha\rangle = \alpha|\alpha\rangle\right)$

no longer a matrix element; just a number...

then

$$= \int dx_{N-1} \dots dx_1 \int dP_{N-1} \dots dP_0 \; \frac{1}{\sqrt{2\pi}} e^{iP_{N-1}x'} e^{-i\frac{P_{N-1}^2}{2m}\left(\frac{T}{N}\right)} e^{-iV(x_{N-1})\left(\frac{T}{N}\right)} \cdot \frac{1}{\sqrt{2\pi}} e^{-iP_{N-1}x_{N-1}} \dots$$

keeps repeating

$$= \int dx_{N-1} \dots dx_1 \int dP_{N-1} \dots dP_0 \; \frac{1}{2\pi} e^{iP_{N-1}(x' - x_{N-1})} e^{-i\frac{P_{N-1}^2}{2m}\frac{T}{N}} e^{-iV(x_{N-1})\frac{T}{N}} \dots$$

Forget potential term $\left(e^{-iV(x_{N-1})\frac{T}{N}}\right)$ and focus on momentum term:

$$K(x',x,T) = \int dP_{N-1}\ \frac{1}{2\pi}\ \exp\left\{-i\frac{P_{N-1}^2}{2m}\frac{T}{N} + iP_{N-1}(x'-x_{N-1})\right\}$$

How do we do this? Analytic Continuation (Wick Rotation, c-time)

$\left(\Delta\tau \equiv \frac{\tau}{N} \quad T \equiv -i\tau\right)$

$$= \int_{-\infty}^{\infty} dP_{N-1}\ \underbrace{\frac{1}{2\pi}\exp\left\{-\Delta\tau\frac{P_{N-1}^2}{2m}\right.}_{\sim \text{ Gaussian! form!}} \left. + iP_{N-1}(x'-x_{N-1})\right\}$$

$$= \int_{-\infty}^{\infty} dP_{N-1}\ \frac{1}{2\pi}\ \exp\left\{-\frac{\Delta\tau}{2m}\left(P_{N-1}^2 - \frac{i\,2m}{\Delta\tau}P_{N-1}(x'-x_{N-1})\right)\right\}$$

And complete the square in P_{N-1}:

$$= \int_{-\infty}^{\infty} dP_{N-1}\ \frac{1}{2\pi}\ \exp\left\{-\frac{\Delta\tau}{2m}\left(P_{N-1} - \frac{im}{\Delta\tau}(x'-x_{N-1})\right)^2 - \frac{m}{2\Delta\tau}(x'-x_{N-1})^2\right\}$$

$$= \frac{1}{2\pi}\exp\left\{-\frac{m}{2}\frac{(x'-x_{N-1})^2}{\Delta\tau}\right\} \underbrace{\int_{-\infty}^{\infty} du\ e^{-u^2}}_{\sqrt{\pi}\ \smile} \sqrt{\frac{2m}{\Delta\tau}}$$

$u \equiv \sqrt{\frac{\Delta\tau}{2m}}\left(P_{N-1} - \frac{im}{\Delta\tau}(x'-x_{N-1})\right)$

$du = dP_{N-1}\sqrt{\frac{\Delta\tau}{2m}}$

$$= \sqrt{\frac{m}{2\pi\Delta\tau}}\ \exp\left\{-\frac{m}{2}\frac{(x'-x_{N-1})^2}{\Delta\tau}\right\}$$

Which repeats N times for $\int dP_{N-1}\dots dP_0$, where we may also re-introduce our potential term

$$= \int dx_{N-1}\cdots dx_1 \int dP_{N-2}\dots dP_0\ \sqrt{\frac{m}{2\pi\Delta\tau}}\ \exp\left\{-\frac{m}{2}\frac{(x'-x_{N-1})^2}{\Delta\tau}\right\}$$

$$\exp\left\{-iV(x_{N-1})\Delta\tau\right\}\ \langle x_{N-1}|P_{N-2}\rangle \dots$$

And take out all following N-2 integrals:

$$= \int dx_{N-1}\cdots dx_1 \left(\frac{m}{2\pi\Delta\tau}\right)^{N/2} \underbrace{\exp\left\{\Delta\tau\left(-\frac{m}{2}\frac{(x'-x_{N-1})^2}{\Delta\tau^2} - V(x_{N-1})\right)\right\}}_{\text{exp. repeats for } x_{N-1},\,\dots,\,x_1}\cdots$$

And re-subbing $\Delta\tau = i\Delta t$ for $\Delta\tau$:

$$= \int dx_{N-1}\cdots dx_1 \left(\frac{m}{2\pi i\Delta t}\right)^{N/2} \exp\left\{i\Delta t\left[\frac{m}{2}\frac{(x'-x_{N-1})^2}{\Delta t^2} - V(x_{N-1})\right]\right\}\cdots$$

which is just the discretized Lagrangian (taking $N\to\infty$)

$$\langle x'|e^{-iHT}|x\rangle = \lim_{N\to\infty}\left(\frac{m}{2\pi i\Delta t}\right)^{N/2}\int\prod_{k=1}^{N-1}\exp\left\{i\Delta t\left[\frac{m}{2}\frac{(x_{k+1}-x_k)^2}{\Delta t^2} - V(x_k)\right]\right\}dx_k$$

Feynman Path Integral

which is the Feynman Path Integral.

'N' determines the number of Paths.

We desire paths which minimize the action, so taking $N\to\infty$ paths recovers the most likely path the particle would take, since $\langle x'|e^{-iHT}|x\rangle$ is a (probability) matrix element between states $|x\rangle \to |x'\rangle$.